BUCKLING ANALYSIS OF A THIN-WALLED COLUMN WITH GENERAL UNSYMMETRICAL OPEN SECTION UNDER A COMBINATION OF BENDING AND TWISTING

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Abstract: **This study investigated buckling of thin-wall columns with general unsymmetrical open section under a combination of bending and twisting. It formulated the equation for torsional-flexural buckling analysis of thin-walled column with general unsymmetrical open section using the principle of Ritz energy formulations. The analysis considers thin-walled columns with open cross-sections of arbitrary slope. During the buckling process, the deformation is assumed to involve a combination of twisting and bending around two axes of the cross-section. The total potential energy functional was derived as the sum of the strain energy functional, U and the potential energy due to the external compressive load, V. The governing equations were found to reduce to a system of algebraic eigenvalue – eigenvector problem. The buckling equation was determined for general asymmetric column sections. The buckling modes were evaluated as flexural torsional buckling modes. Generally, the critical buckling loads decreased as the length of the column increased. The findings are in excellent agreement with Jerath (2020)'s solution for thin-walled columns with general nnsymmetrical open section, and with hinged (simply supported) conditions at the ends x = 0, and x = L. The buckling mode was found to be a cubic polynomial.**

Keywords: **Unsymmetricalcolumns, flexural torsional buckling mode, algebraic eigen vector problem, cubic polynomial.**

1. INTRODUCTION

A thin-wall column refers to a structural element that is slender in shape, typically with a height much greater than its cross-sectional dimensions. Thin-walled open section has garnered considerable interest across diverse engineering applications, spanning from automotive design to the development of aeronautical components (Barszcz et al., 2022). For different applications, thin-walled elements can be relatively easily formed into open or closed, singly or doubly symmetric cross-sections. They provide good strength and stiffness with a relatively small amount of material. Designers have been taking advantage of this weight efficiency in various applications where self-weight of the structure, is a key design factor.

In recent years, there has been a growing trend in the use of very slender thin-walled cross-sections members due to their high stiffness/weight ratio, as noted by Simao and Simoes da Silva (2004). Various industries, including civil, mechanical, naval, and aerospace, have been seeking stronger and lighter structural solutions to optimize effectiveness and cost, as highlighted by Andreassen (2012).

Buckling is the phenomenon where a structure suddenly shifts from one equilibrium state to another under load (Possidente et al., 2020). The load level at which this transition happens is known as the buckling load. The behavior of a standard thin-walled composite structure, can be described in three major states: pre-buckling, critical buckling, and post-buckling. The loading process and carrying capacity of the structure are determined by buckling characteristics. The structure can still function even after buckling, because the post-critical equilibrium trend is stable, and the increasing compressive load increases the wall deflection (Rozylo et al., 2020)

Monosymmetric sections and asymmetric sections are susceptible to both torsional and flexural buckling modes, and sometimes these modes may occur at a lower load than flexural buckling about the minor axis. In bridges, these buckling modes are relevant primarily to angle and channel bracing members. Hendy and Murphy (2007) gave a comprehensive discussion on torsional modes.

A number of analytical solutions were developed on the basis of the Vlasov theory of thin-walled members (Barszcz et al., 2022). More recent research on the theory of flexural-torsional buckling has been presented by Tong and Zhang (2003a) and (2003b) with their investigations of a new theory to clarify the inconsistencies of existing theories of the flexural-torsional buckling of thin-walled members. Ezeh (2009) conducted a theoretical analysis based on Vlasov's theory, as modified by Varbanov, to examine flexural, flexural-torsional, and flexural-torsional-distortional buckling modes of thin-walled closed columns. Chidolue and Osadebe (2012) also employed Vlasov's theory for the torsional-distortional analysis of thin-walled box girder bridges. Similarly, Chidolue and Aginam (2012) studied the effects of shape factor on the flexural-torsional-distortional behavior of thin-walled box girder structures using Vlasov's theory. In another study, Ezeh (2010) investigated the buckling behavior of axially compressed multi-cell doubly symmetric thin-walled columns using Vlasov's theory. Additional works by Osadebe and Chidolue (2012a, 2012b), and Osadebe and Ezeh (2009a, 2009b) were also grounded in Vlasov's method. Furthermore, Ezeh and Osadebe (2010) conducted a comparative study on Vlasov and Euler instabilities of axially compressed thin-walled box columns.

The use of energy methods in buckling problems is widely studied by Gizejowski et al. (2021). Onyechere et al. (2021), used orthogonal polynomial displacement expressions in the dynamic analysis of thick plates. According to Barszcz et al. (2023), The strain-energy stored in an open-section member results from two non zero strain components, normal and shear.

Since thin-walled structures tend to be slender, they are vulnerable to buckling instabilities at the local as well as the global scales. The main motivation for this study is the need to derive simplified stability equation for flexural-torsional (FT) buckling analysis of thin-walled columns of open sections. Therefore this research aimed to achieve the following objectives:

i. To determine the total potential energy functional of a general unsymmetrical thin-walled column with open cross-section undergoing flexural-torsional buckling.

ii. To obtain the differential equation for the flexural-torsional buckling analysis of general unsymmetrical thin-walled columns with open cross-sections.

iii. To obtain elastic buckling equation using energy formulation for General Unsymmetrical thin-walled columns with open cross-sections.

iv. To solve numerical problems with the method developed herein

2. THEORETICAL BACKGROUND

2.1 Basic Assumptions.

The formulation of the potential energy functional is based on the following assumptions:

(i) Shear centre of the cross-section is chosen as the origin.

(ii) The x and y coordinate axes are assumed to be coincident with the principal axes of the open cross-section, and the z coordinate axis is the longitudinal axis of the thin-walled column through the shear centre.

(iii) The displacement field include the displacements along x, y and z direction designated as u, v and w. Strain energy and external work for each case shall be treated independently first. The positions of O and S stand for the centroid and shear center respectively.

(iv) The displacement of the shear center along y-axis and z- axis is denoted as v and w respectively. On the other hand, the displacement of the centroid along y-axis and z-axis are denoted as v*and w*.

(v) The linear space between the shear center and the centroid remains constant after the translational displacement.

(vi) The shear centre of the cross-section is chosen as the origin. The x and y coordinate axes are assumed to be coincident with the principal axes of the open cross-section, and the z coordinate axis is the longitudinal axis of the thin-walled column through the shear centre.

2.2 Determination of the Total Potential Energy Functional for Thin-Walled Column with Open Cross-Section undergoing Flexural-Torsional Buckling

The total potential energy functional ($\prod$) for the thin-walled column with open cross-section under flexural-torsional buckling is the sum of the strain energy functional, U and the potential energy, V due to the external compressive load as given by Equation (1).

$$\Pi = U - V \tag{1}$$

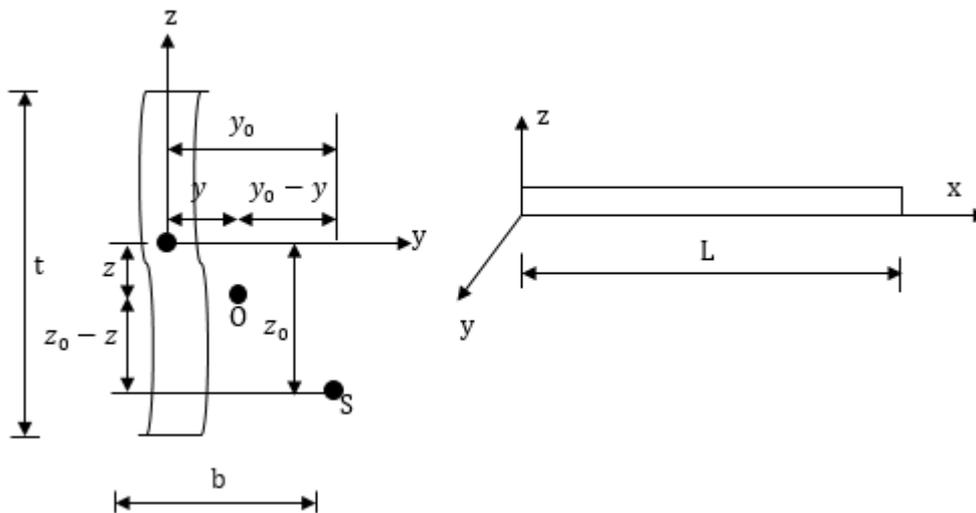Consider a column with arbitrary cross section as shown on Figure 1.

Figure 1: Column under axial load

This analysis took into consideration column buckling under the following cases:

i.   Pure flexural buckling

ii. Pure torsional buckling

iii. Flexure - torsional buckling

$$w^* = w - (y_0 - y).\emptyset \tag{2}$$

$$v^* = v + (z_0 - z).\emptyset \tag{3}$$

Where;$\emptyset$ represents the rotation of the cross-section about the shear is center O, y_0 and z_0 represent the coordinates of the shear center O

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2.3 Flexural Buckling

Let a portion of the column be considered. The total change in length Δ_z of the column after buckling is as given by Equation (4).

That is:

$$\Delta_z = \frac{1}{2} \int_0^L \left(\frac{\partial w^*}{\partial x} \right)^2 dx \tag{4}$$

Similarly, for buckling in the y direction, the change in length Δ_y, is given by Equation 4:

$$\Delta_y = \frac{1}{2} \int_0^L \left(\frac{dv^*}{dx} \right)^2 dx \tag{5}$$

Total buckling is obtained by adding the buckling in both y and z directions. That is adding Equations (1) and (5) as follows:

$$\Delta = \frac{1}{2} \int_0^L \left[\left(\frac{dv^*}{dx} \right)^2 + \left(\frac{dw^*}{dx} \right)^2 \right] dx \tag{6}$$

The indefinite summation product of axial stress and buckling caused by it within the domain (cross section area of the column), gives the external work:

$$V = \iint_A \sigma_x \Delta \, dA \tag{7}$$

From Kirchhoff's assumptions of zero shear strains:

$$\gamma_{xz} = \frac{du_z}{dz} + \frac{dw}{dx} = 0 \tag{8}$$

$$\gamma_{xy} = \frac{du_y}{dy} + \frac{dv}{dx} = 0 \tag{9}$$

Solving Equations (8) and (9) gives Equation (10) and (11):

$$u_z = -z \frac{dw}{dx} \tag{10}$$

$$u_y = -y \frac{dv}{dx} \tag{11}$$

But, normal strain in x-direction is obtained as the first derivative of Equations (10) and (11) with respect to x:

$$\varepsilon_x{}^z = -z \frac{d^2 w}{dx^2} \tag{12}$$

$$\varepsilon_x{}^z = -y \frac{d^2 v}{dx^2} \tag{13}$$

Adding Equations (12) and (13) gives the normal strain ε_x, along x-axis as:

$$\varepsilon_x = -\left(z \frac{d^2 w}{dx^2} + y \frac{d^2 v}{dx^2} \right) \tag{14}$$

From Hooke's law, stress σ_x, is mathematically defined as:

$$\sigma_x = E \varepsilon_x = -E \left(z \frac{d^2 w}{dx^2} + y \frac{d^2 v}{dx^2} \right) \tag{15}$$

Average strain energy is given bt Equation (16):

$$U = \frac{E}{2} \int \left[I_z \left(\frac{d^2 w}{dx^2} \right)^2 + 2 I_{yz} \frac{d^2 w}{dx^2} * \frac{d^2 v}{dx^2} + I_y \left(\frac{d^2 v}{dx^2} \right)^2 \right] dx \tag{16}$$

Where:

$$I_{yz} = \int_{b_1}^{b_2} \int_{t_1}^{t_2} yz \, dydz = 0 \tag{17}$$

Substituting Equation (17) into Equation (16) , yields Equation (18a):

$$U = \frac{E}{2} \int \left[I_z \left(\frac{d^2 w}{dx^2} \right)^2 + I_y \left(\frac{d^2 v}{dx^2} \right)^2 \right] dx \tag{18a}$$

Substituting Equation (15) into Equation (7), gives:

$$V = \iint_A \sigma_x \frac{1}{2} \int_0^L \left[\left(\frac{dv^*}{dx} \right)^2 + \left(\frac{dw^*}{dx} \right)^2 \right] dx \, dA \tag{18b}$$

Where A is cross sectional area

That is:

$$V = \frac{\sigma_x}{2} \iint_A \int_0^L \left(\left[\frac{dv}{dx} \right]^2 + \left[\frac{dw}{dx} \right]^2 + 2[z_0 - z].\frac{dv}{dx}\frac{d\phi}{dx} - 2[y_0 - y].\frac{dw}{dx}\frac{d\phi}{dx} + \left[\begin{matrix} y_0{}^2 - 2y_0 y \\ +y^2 + z_0{}^2 - 2z_0 z + z^2 \end{matrix} \right].\left[\frac{d\phi}{dx} \right]^2 \right) dx \, dA \tag{19}$$

If a column section is symmetrical about two axes, the shear center coincides with the centroid, and therefore $y_0 = z_0 = 0$. This implies that:

$$V = \frac{\sigma_x A}{2} \int_0^L \left(\left[\frac{dv}{dx} \right]^2 + \left[\frac{dw}{dx} \right]^2 + \left[y_0{}^2 + z_0{}^2 + \frac{I_y + I_z}{A} \right].\left[\frac{d\phi}{dx} \right]^2 - 2y_0.\frac{dw}{dx}\frac{d\phi}{dx} + 2z_0.\frac{dv}{dx}\frac{d\phi}{dx} \right) dx \tag{20}$$

Where:

$$I_y = \iint_A y^2 \, dAn \tag{21}$$

$$I_z = \iint_A z^2 \, dA \tag{22}$$

$$0 = \iint_A y \, dA = \iint_A z \, dA \tag{23}$$

Since the case of pure flexural buckling is considered, the torsional work is ignored. Thus, Equation (20) becomes:

$$V = \frac{N_x}{2} \int_0^L \left[\left[\frac{dv}{dx} \right]^2 + \left[\frac{dw}{dx} \right]^2 \right] dx \tag{24}$$

where:

$$I_0 = Ay_0{}^2 + Az_0{}^2 + I_y + I_z \tag{25}$$

$$N_x = \sigma_x A \tag{26}$$

Total potential energy is the algebraic summation of strain energy and external work:

$$\Pi = U - V \tag{27}$$

Substituting Equation 20 and 24 into Equation 27 gives:

$$\Pi = \frac{EI_z}{2} \int \left(\frac{d^2 w}{dx^2} \right)^2 dx + \frac{EI_y}{2} \int \left(\frac{d^2 v}{dx^2} \right)^2 dx - \frac{N_x}{2} \int_0^L \left(\frac{dw}{dx} \right)^2 dx - \frac{N_x}{2} \int_0^L \left(\frac{dv}{dx} \right)^2 dx \tag{28}$$

2.4 Total Potential Energy Functional for Thin-Walled Column of Open Cross-Section In Torsional Buckling

This case has two sub cases. Case A is is one in which, ends are allowed to warp, and sub the case B, deals with where ends, are prevented from warping.

2.4.1 Case A: End Free to Warp

Assume the ends of the column in Figure 3.1 are allowed to warp as illustrated on Figure 3.2:

From similar triangles:

$$\frac{v_1}{(t_1)} = \frac{v_2}{(t_2)} = \tan\phi \tag{29}$$

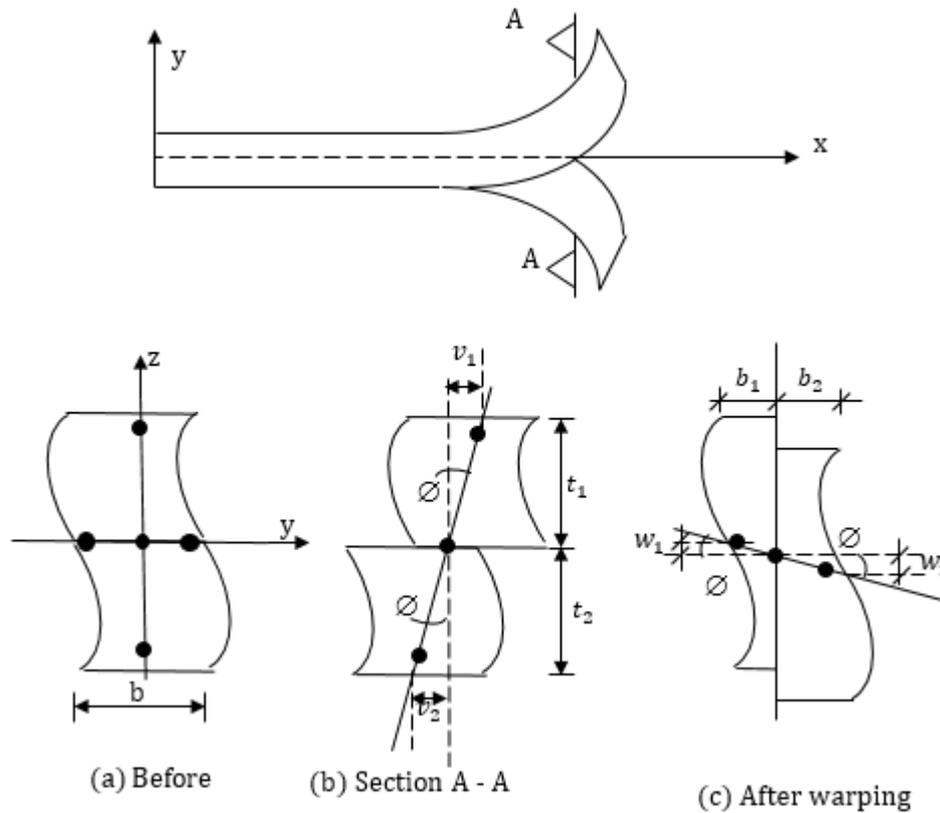$$\frac{w_1}{(b_1)} = \frac{w_2}{(b_2)} = \tan\phi \tag{30}$$

Figure 2: One end of column warped

For small deformations,

$$\phi = \tan \phi \tag{31}$$

Substituting Equation (31) into Equations (29) and (30) gives:

$$v_i = t_i\, \phi \tag{32}$$

$$w_i = b_i\, \phi \tag{33}$$

Equations (32) and (33) are rewritten as:

$$v = z\, \phi \tag{34}$$

$$w = y\, \phi \tag{35}$$

From Kirchhoff's assumptions of zero shear strains, shear deformation, γ_{xy} is equal to zero

$$\gamma_{xy} = \frac{du_y}{dy} + \frac{dv}{dx} = 0 \tag{36}$$

Solving Equation (36), g yields Equation (37):

$$u_y = -y\frac{dv}{dx} \tag{37}$$

Substituting Equation (32) into Equation (37) gives:

$$u_y = -y\, t_i\frac{d\phi}{dx} \tag{38}$$

Similarly if the warping of the flanges moves in z direction,

$$u_z = -z b_i\frac{d\phi}{dx} \tag{39}$$

Axial displacement u, is the summation of Equations (38) and (39). That is:

$$u = u_y + u_z = -y\, t_i \frac{d\phi}{dx} - zb_i \frac{d\phi}{dx} = -(y\, t_i + zb_i)\frac{d\phi}{dx} \tag{40}$$

Normal strain ε_x in x direction is the first derivative of Equations (40) with respect to x:

$$\varepsilon_x = -(y\, t_i + zb_i)\frac{d^2\phi}{dx^2} \tag{41}$$

The twist (shear strain around x-axis), is obtained by adding together, the first derivatives of w and v with respect to x. That is the summation of the first derivatives of Equations (34) and (35) with respect to x. That is:

$$\gamma_s = \varepsilon_{zx} + \varepsilon_{yx} \tag{42}$$

Where γ_s is shear strain around x-axis

The first derivatives of Equation (34) and (35) with respect to x are:

$$\varepsilon_{yx} = \frac{dv}{dx} = z\frac{d\phi}{dx} \tag{43}$$

$$\varepsilon_{zx} = \frac{dw}{dx} = y\frac{d\phi}{dx} \tag{44}$$

Substituting Equations (43) and (44) into (42) yields:

$$\gamma_s = (y + z)\frac{d\phi}{dx} \tag{45}$$

From Hooke's law, normal and shear stresses are mathematically defined by Equation (46) and (47) respectively;

$$\sigma_x = E\varepsilon_x = -E(y\, t_i + zb_i)\frac{d^2\phi}{dx^2} \tag{46}$$

$$\tau_s = G\gamma_s = G(y + z)\frac{d\phi}{dx} \tag{47}$$

where τ_s is shear stress

Average strain energy U_y is given by Equation (48)

$$U_y = \frac{1}{2}\int\int_{b_1}^{b_2}\int_{t_1}^{t_2}(\sigma_x\varepsilon_x + \tau_s\gamma_s)\,dxdydz \tag{48}$$

Substituting equations (41), (42), (45) and (46) into equation (48) gives:

$$U = \frac{EI_\omega}{2}\int\left(\frac{d^2\phi}{dx^2}\right)^2 dx + \frac{GJ}{2}\int\left(\frac{d\phi}{dx}\right)^2 dx \tag{49}$$

Where: the warping torsional constant I_w and St. Venant torsional constant J are defined as:

$$I_\omega = \int_{b_1}^{b_2}\int_{t_1}^{t_2}(y\, t_i + zb_i)^2\,dydz \tag{50}$$

$$J = \int_{b_1}^{b_2}\int_{t_1}^{t_2}(y + z)^2\,dydz \tag{51}$$

From Equation (11), the ratio, $\left(\frac{u}{y}\right)$ squared is obtained

$$\left(\frac{u}{y}\right)^2 = \left(\frac{dv}{dx}\right)^2 \tag{52}$$

Substituting Equation (52) into Equation (20), yields Equation (53):

$$V_y = \frac{1}{2}\int_0^L\int_{t_1}^{t_2} N_x\left(\frac{u}{y}\right)^2 dxdz \tag{53}$$

Rearranging Equation (38) gives:

$$\frac{u}{y} = -t_i\frac{d\phi}{dx} \tag{54}$$

Substituting Equation (54) into Equation (53), yields Equation (55)

$$V_y = \frac{N}{2} \frac{I_0}{A_c} \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \qquad (55)$$

where:

$$\frac{NI_0}{A_c} = \int_{t_1}^{t_2} N_x t_i^2 \, dz \qquad (56)$$

where: A_c is the area of the cross section.

I_o is the polar moment of inertia of the cross-section about the longitudinal axis passing through the shear center O.

Adding together equation (49) and (55) gives the total potential energy, Π:

$$\Pi = \frac{EI_\omega}{2} \int \left(\frac{d^2\phi}{dx^2}\right)^2 dx - \frac{N}{2} \frac{I_0}{A_c} \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \qquad (57)$$

2.4.2   Case B: Ends Prevented from Warp

This is a case where only twisting is applied with ends prevented from warping. Consider, for instance if the column in Figure 1 is assumed to be circular bar as shown on Figure 3.
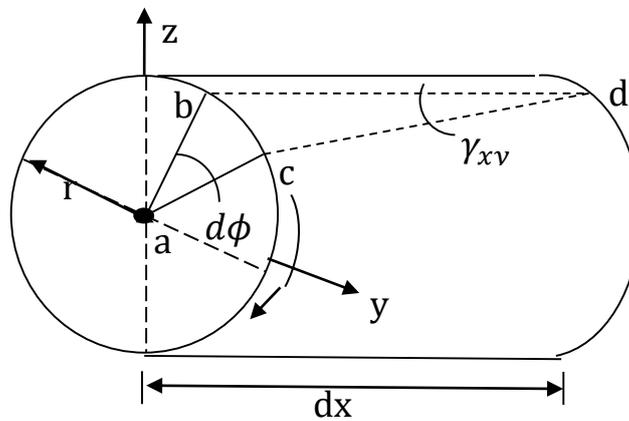

Figure 3: Small length of a circular shaft undergoing twisting

Since the deformation is small, the angles (twist and shear strain) are defined by Equations (58) and (59) respectively:

$$d\phi = \sin d\phi = \frac{bc}{ac} = \frac{bc}{r} \qquad (58)$$

$$\gamma_{xy} = \sin \gamma_{xy} = \frac{bc}{bd} = \frac{bc}{dx} \qquad (59)$$

From Equations (58) and (59), it is gathered that:

$$\gamma_{xy} = r\frac{d\phi}{dx} \qquad (60)$$

Now, consider r to be y. This makes Equation (60) to become Equation (61):

$$\gamma_{xy} = y\frac{d\phi}{dx} \qquad (61)$$

On the other hand let r be z. This then makes Equation (60) to become:

$$\gamma_{xz} = z\frac{d\phi}{dx} \qquad (62)$$

From Hooke's law, shear stress, τ_{xy}, is mathematically defined as:

$$\tau_{xy} = G\gamma_{xy} \qquad (63)$$

Substituting Equation (61) into Equation (63) gives:

$$\tau_{xy} = G.y.\frac{d\phi}{dx} \tag{64}$$

Where G is shear modulus

Strain energy U_{xy}, is defined as the product of stress, strain and volume of matter. Thus, average strain energy is given by Equation (65):

$$U_{xy} = \frac{1}{2}\int\int\left[\int \tau_{xy}.\gamma_{xy}\,dy\right]dx\,dz \tag{65}$$

Substituting Equations (61) and (64) into Equation (65) gives Equation (66):

$$U_{xy} = \frac{GJ}{2}\int\left(\frac{d\phi}{dx}\right)^2 dx \tag{66}$$

Where J is the St. Venant torsional constant defined as:

$$J = \int\int y^2\,dydz \tag{67}$$

The total potential energy of a column subject to torsional buckling is obtained by adding together Equations (57) and (66):

$$\Pi = \frac{EI_\omega}{2}\int\left(\frac{d^2\phi}{dx^2}\right)^2 dx - \frac{N_x}{2}\frac{I_0}{A_c}\int_0^L\left(\frac{d\phi}{dx}\right)^2 dx + \frac{GJ}{2}\int\left(\frac{d\phi}{dx}\right)^2 dx \tag{68}$$

2.5 Total Potential Energy Functional for Thin-Walled Column of Open Cross-Section in Flexural-Torsional Buckling

The strain energy U for this case, is obtained by adding together Equations (18), (49) and (66):

$$U = \frac{EI_z}{2}\int\left(\frac{d^2w}{dx^2}\right)^2 dx + \frac{EI_y}{2}\int\left(\frac{d^2v}{dx^2}\right)^2 dx + \frac{EI_\omega}{2}\int\left(\frac{d^2\phi}{dx^2}\right)^2 dx + \frac{GJ}{2}\int\left(\frac{d\phi}{dx}\right)^2 dx \text{ v} \tag{69}$$

where:

$$I_z = \iint_A z^2\,dA \tag{70}$$

$$I_y = \iint_A y^2\,dA \tag{71}$$

$$I_\omega = \iint_A (y\,t_i + zb_i)^2\,dA \tag{72}$$

$$J = \iint_A (y+z)^2\,dA \tag{73}$$

Subtracting Equation (24) from Equation (69) gives the total potential energy functional, Π:

$$\Pi = \frac{E}{2}\int\left[I_z\left(\frac{d^2w}{dx^2}\right)^2 + I_y\left(\frac{d^2v}{dx^2}\right)^2 + I_\omega\left(\frac{d^2\phi}{dx^2}\right)^2 + \frac{GJ}{E}\left(\frac{d\phi}{dx}\right)^2 - \frac{N_x}{E}\left(\frac{dv}{dx}\right)^2 - \frac{N_x}{E}\left(\frac{dw}{dx}\right)^2 - \frac{N_xI_0}{EA}.\left(\frac{d\emptyset}{dx}\right)^2 + \frac{2y_0N_x}{E}.\frac{dw}{dx}.\frac{d\emptyset}{dx}\right.$$
$$\left. - \frac{2z_0N_x}{E}.\frac{dv}{dx}.\frac{d\emptyset}{dx}\right]dx \tag{74a}$$

where:

$$I_0 = Ay_0^2 + Az_0^2 + I_y + I_z \tag{74b}$$

Equation (74a) can be written in non-dimensional coordinate, R (where: R = x/L) as:

$$\Pi = \frac{E}{2L^3}\int\left[I_z\left(\frac{d^2w}{dR^2}\right)^2 + I_y\left(\frac{d^2v}{dR^2}\right)^2 + I_\omega\left(\frac{d^2\phi}{dR^2}\right)^2 + \frac{GJL^2}{E}\left(\frac{d\phi}{dR}\right)^2 - \frac{N_xL^2}{E}\left(\frac{dv}{dR}\right)^2 - \frac{N_xL^2}{E}\left(\frac{dw}{dR}\right)^2 - \frac{N_xI_0L^2}{EA}.\left(\frac{d\emptyset}{dR}\right)^2\right.$$
$$\left. + \frac{2y_0N_xL^2}{E}.\frac{dw}{dR}.\frac{d\emptyset}{dR} - \frac{2z_0N_xL^2}{E}.\frac{dv}{dR}.\frac{d\emptyset}{dR}\right]dR \tag{74c}$$

2.6 Determination of the Differential Equation for the Flexural Torsional Buckling Analysis of Thin-Walled Columns with Open Cross-Sections

The differential equations was obtained by minimizing the total potential energy functional with respect to the displacement functions. v, w and ϕ.

Minimizing Equation (74c) with respect to v gives:

$$\frac{I_y E}{N_x L^2}\left(\frac{d^4 v}{dR^4}\right) - \frac{d^2 v}{dR^2} - z_0 \cdot \frac{d^2 \phi}{dR^2} = 0 \qquad (75)$$

Minimizing Equation (74c) with respect to w gives:

$$\frac{I_z E}{N_x L^2}\frac{d^4 w}{dR^4} + \frac{d^2 w}{dR^2} + y_0 \cdot \frac{d^2 \phi}{dR^2} = 0 \qquad (76)$$

Minimizing Equation (74c) with respect to ϕ gives:

$$\frac{I_\omega E}{N_x L^2}\frac{d^4 \phi}{dR^4} + \left[\frac{GJ}{N_x} - \frac{I_0}{A}\right]\frac{d^2 \phi}{dR^2} + y_0 \cdot \frac{d^2 w}{dR^2} - z_0 \cdot \frac{d^2 v}{dR^2} = 0 \qquad (77)$$

Solving equations (75). (76) and (77) simultaneously gave the following relations:

$$\phi = g_1 . v \qquad (78)$$

$$\phi = g_2 . w \qquad (79)$$

$$v = g_3 . w \qquad (80)$$

Where: g_1, g_2 and g_3 are constants that will be determined later.

Substituting Equations (79) and (80) into Equation (74), yields:

$$\Pi = \frac{EI_T}{2L^3}\int\left[\left(\frac{d^2 w}{dR^2}\right)^2 + \left(\frac{dw}{dR}\right)^2 \cdot \frac{N_T L^2}{EI_T}\right]dR \qquad (81)$$

where:

$$I_T = I_z + I_y . g_3{}^2 + I_\omega . g_2{}^2 \qquad (82)$$

$$N_T = N_x\left(\frac{GJ}{N_x}. g_2{}^2 - g_3{}^2 - 1 - \frac{I_0}{A}. g_2{}^2 + 2. g_2 y_0 - 2. g_2 g_3 z_0\right) \qquad (83)$$

Minimizing Equation (81) with respect to w, gives the governing equation of the thin-walled open cross section column undergoing flexural-torsional buckling as:

$$\frac{d\Pi}{dw} = \frac{EI_T}{L^3}\int\left[\frac{d^4 w}{dR^4} + \frac{d^2 w}{dR^2} \cdot \frac{N_T L^2}{EI_T}\right]dR = 0 \qquad (84)$$

For Equation (84) to be true, its integrand must be zero. That is:

$$\frac{d^4 w}{dR^4} + \frac{d^2 w}{dR^2} \cdot \frac{N_T L^2}{EI_T} = 0 \qquad (85)$$

2.7 Determination of Buckling Load Formulae

The ready solution for Equation (85) is given by Equation (86):

$$w = hB_2 \qquad (86)$$

Where:

$$B_2 = [a_1 a_2 a_3 a_4]^T \qquad (87)$$

$$h = [1 \ R \ cos \, BR \ sin \, BR] \qquad (88)$$

Substituting Equation (86) into Equations (78) and (80) gave respectively Equation (89) and (90):

$$B_3 = g_2 . B_2 \qquad (89)$$

$$B_1 = g_3 . B_2 \qquad (90)$$

Substituting 86 into Equation (81), yields:

$$\Pi = \frac{E}{2L^3} \int \left[\left(\frac{d^2h}{dR^2}\right)^2 \left(I_z.B_2{}^2 + I_y.g_3{}^2B_2{}^2 + I_\omega.g_2{}^2B_2{}^2\right) + \left(\frac{dh}{dR}\right)^2 \left(\frac{GJ}{N_x}.g_2{}^2B_2{}^2 - g_3{}^2B_2{}^2 - B_2{}^2 - \frac{I_0}{A}.g_2{}^2B_2{}^2 + \right. \right.$$

$$\left. \left. 2.g_2B_2{}^2y_0 - 2.g_2g_3B_2{}^2z_0\right).\frac{N_xL^2}{E}\right] dR \tag{91}$$

Substituting Equations (89) and (90) into Equation (91) gives Equation (92):

$$\Pi = \frac{E}{2L^3}\left(B_1{}^2\left[I_y k_{RR} - \frac{N_xL^2}{E}k_R\right] + B_2{}^2\left[I_z k_{RR} - \frac{N_xL^2}{E}k_R\right] + B_3{}^2\left[I_\omega k_{RR} + \frac{L^2}{E}GJk_R - \frac{N_xL^2}{E}.\frac{I_0}{A}k_R\right] + 2B_2B_3y_0.\frac{N_xL^2}{E}k_R - 2B_1B_3z_0.\frac{N_xL^2}{E}k_R\right) \tag{92}$$

2.7.1 General Case of Unsymmetrical Open Section

Minimizing Equation (92) with respect to B_1 gives:

$$B_1\left[\frac{EI_y}{L^2}\frac{k_{RR}}{k_R} - N_x\right] - B_3z_0.N_x = 0 \tag{93}$$

Minimizing Equation (92) with respect to B_2 gives:

$$B_2\left[\frac{EI_z}{L^2}\frac{k_{RR}}{k_R} - N_x\right] + B_3y_0.N_x = 0 \tag{94}$$

Minimizing Equation (92) with respect to B_3 gives:

$$B_3\left[\frac{EI_\omega}{L^2}\frac{k_{RR}}{k_R} + GJ - N_x.\frac{I_0}{A}\right] + B_2y_0.N_x - B_1z_0.N_x = 0 \tag{95}$$

For pure flexural buckling case, the flexural critical buckling load is defined from Equations (93), (94) and (95) as:

$$N_c = \frac{EI}{L^2}.\frac{k_{RR}}{k_R} \tag{96}$$

where: I can be either I_y, I_z or I_ω as the case may be

The Equation (96) is rewritten for various second moments of area as seen in Equation (97), (98) and (99):

$$N_{cy} = \frac{EI_y}{L^2}.\frac{k_{RR}}{k_R} \tag{97}$$

$$N_{cz} = \frac{EI_z}{L^2}.\frac{k_{RR}}{k_R} \tag{98}$$

$$N_{c\omega} = \frac{EI_\omega}{L^2}.\frac{k_{RR}}{k_R} \tag{99}$$

Substituting Equations (97), (98) and (99) into Equations (93), (94) and (95) respectively gives:

$$B_1[N_{cy} - N_x] - B_3z_0.N_x = 0 \tag{100}$$

$$B_2[N_{cz} - N_x] + B_3y_0.N_x = 0 \tag{101}$$

$$B_3[N_\emptyset - N_x.]\frac{I_0}{A} + B_2y_0.N_x - B_1z_0.N_x = 0 \tag{102}$$

where:

$$N_\emptyset = (N_{c\omega} + GJ)\frac{A}{I_0} \tag{103}$$

$$N_{c\omega} = \frac{EI_\omega}{L^2}.\frac{k_{RR}}{k_R}; \qquad I_\omega = I_{yz} = I_y.h_i{}^2 + I_z.b_i{}^2 = warping \ constant$$

I_z is the second moment of area about (around) y axis and I_y is the second moment of area about (around) z axis.

The Equations (100), (101) and (102) can be put in matrix forms as:

$$\begin{bmatrix} N_{cy} - N_x & 0 & -z_0.N_x \\ 0 & N_{cz} - N_x & y_0.N_x \\ -z_0.N_x & y_0.N_x & [N_\emptyset - N_x.]\frac{I_0}{A} \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = 0 \tag{104}$$

For non-trivial solution, the determinant of Equation (104) must be zero. That is:

$$\begin{vmatrix} N_{cy} - N_x & 0 & -z_0 . N_x \\ 0 & N_{cz} - N_x & y_0 . N_x \\ -z_0 . N_x & y_0 . N_x & [N_\emptyset - N_x .] \frac{I_0}{A} \end{vmatrix} = 0 \tag{105}$$

Simplifying the matrix equation gives

$$D_1 . N_x^3 + D_2 . N_x^2 + D_3 . N_x + D_4 = 0 \tag{106}$$

Where:

$$D_1 = 1 - [y_0^2 + z_0^2] \frac{A}{I_0} \tag{107}$$

$$D_2 = [y_0^2 . N_{cy} + z_0^2 . N_{cz}] \frac{A}{I_0} - N_{cy} - N_{cz} - N_\emptyset \tag{108}$$

$$D_3 = N_{cy} N_{cz} + N_{cy} N_\emptyset + N_{cz} N_\emptyset \tag{109}$$

$$D_4 = -N_{cy} N_{cz} N_\emptyset \tag{110}$$

3. NUMERICAL EXAMPLE

Determine the critical buckling loads, Nx, of a generally unsymmetrical unequal angle section with hinged ends. The requirements for the shape functions are that they must satisfy the geometric boundary conditions. The properties of the channel are tabulated as obtained from Steel Designer's Manual (2012).

Section Designation is 200 x 150 x 12; Torsional index = X; Warping torsional constant = I_ω; St. Venant torsional constant = J.

Table 1: Properties of an unsymmetrical unequal angle section

Here is the data presented in a clean table format:

D (mm)	B (mm)	t (mm)	M (kg/m)	A (cm²)	Iz (cm⁴)	Iy (cm⁴)	Iω (cm⁶)	Cy (cm)	Cz (cm)
200	150	12	32	40.8	803	1650	494	6.08	3.61

4. RESULTS

The results of numerical analysis of the problem given as Example case 1, presented in tables 2 and Figure 1

Table 2: Critical buckling load of generally unsymmetric unequal angle section with hinged ends

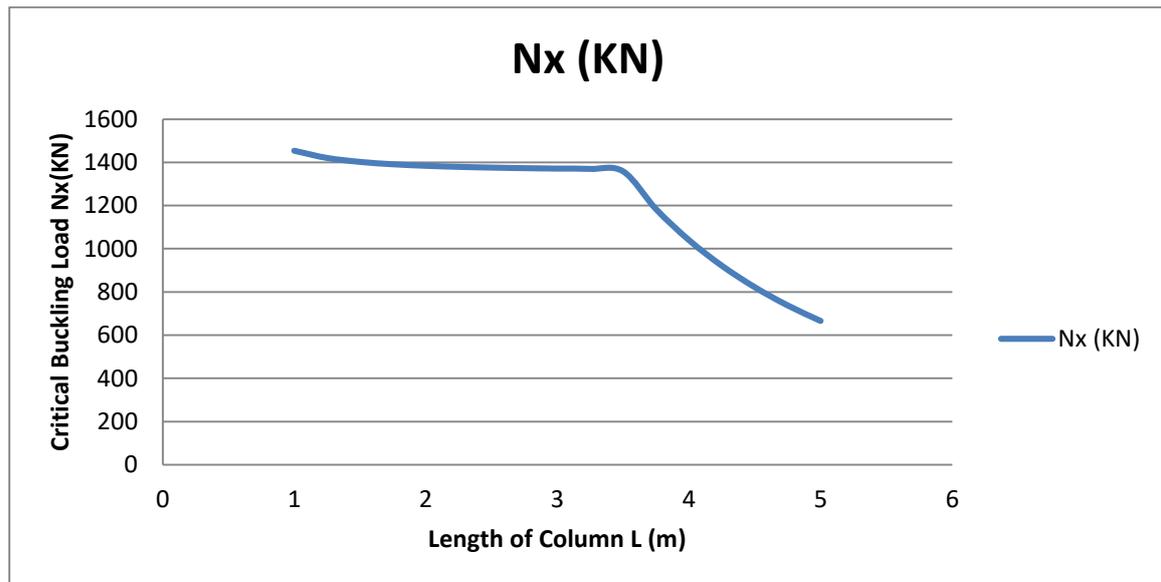| L(m)	Nx (KN)
1	1454.256134
1.25	1420.798386
1.5	1402.623806
1.75	1391.665108
2	1384.552491
2.25	1379.676105
2.5	1376.188054
2.75	1373.607291
3	1371.644409
3.25	1370.116826
3.5	1358.621543
3.75	1183.510322
4	1040.194619
4.25	921.4180707
4.5	821.882168
4.75	737.6449375
5	665.7245561

Figure 1: Graph of Critical buckling against the length of column for general unsymmetric unequal angle section.

5. DISCUSSION

The results of the numerical analysis carried out in this work were presented. In general, the equations were found to be reduced to a system of algebraic eigenvalue eigenvector problem. The buckling equations was found for single general asymmetric column sections. The buckling modes were found as flexural-torsional buckling modes. For general asymmetric sectioned column the buckling mode was found to be a cubic polynomial. Initially, the critical buckling loads , Nx, decreased gradually as the length of the column increased, but subsequently (from a length of 3.25 to be precise) it decrease rapidly until the column buckled.

The results obtained in this present study, were compared with result of other existing research to determine their efficacy. The results of the Example were compared with the corresponding solutions of the same problem solved by Jerath (2020) using differential equations method. The results obtained were about the same.

The result was further compared with the solution obtained by Iyengar (1988), which utilized the equilibrium of the deformed shape approach. Both solutions were found to be approximately the same.

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